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BUCKLING OF A LONG CYLINDRICAL SHELL SURROUNDED BY AN ELASTIC MEDIUM

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Abstract—A long, thin, circular, cylindrical shell is subjected to uniform external pressure exerted by a surrounding elastic medium. The stability of equilibrium of the shell is examined by considering possible neighboring equilibrium states. The loading exerted by the elastic medium on the shell in the deformed state is found by solving an associated boundary value problem of the linearized theory of elasticity in the presence of initial stress. Expressions are derived which give the critical pressure for the cases of a bonded and a smooth interface.

NOMENCLATURE

a	shell radius
$D = Eh^3/12(1-\nu^2)$	shell flexural modulus
E	Young's modulus for the shell
E_j	unit elongation of a line element originally in the j-direction
E_m	Young's modulus of the elastic medium
$E_p = Eh/(1-\nu^2)$	shell compressional modulus
f_θ, f_r	tangential and radial components of boundary traction taken in the undeformed position per unit original area
$\Delta F_\theta, \Delta F_r$	circumferential and radial components of the change due to deformation of the shell surface tractions per unit original area
$\Delta F_\theta^i, \Delta F_r^i$	circumferential and radial components of the change due to deformation of the initial uniform pressure on the shell surface
F_θ^a, F_r^a	additional shear and normal components of stress at the shell medium interface induced by the shell displacements
G, H, L, M, R, S	dimensionless parameters defined by equation (13)
h	shell thickness
K	parameter defined by equation (25)
$\mathbf{k}_r, \mathbf{k}_\theta$	unit vectors tangent to the undeformed coordinate lines
$\mathbf{k}_r', \mathbf{k}_\theta'$	unit vectors tangent to the deformed coordinate lines
n	buckling mode number
p	critical pressure of a long, thin, circular, cylindrical shell surrounded by a uniformly compressed elastic medium
p_c, p_0	critical pressures of long, thin, circular, cylindrical shells subjected to uniform constant-directional and uniform hydrostatic pressures, respectively
r, θ	cylindrical coordinates
u_θ, u_r	tangential and radial displacement components in the elastic medium
v, w	tangential and radial displacement components of the middle surface of the shell measured positive clockwise and radially outward, respectively
Δ	dilatation in the linear theory of elasticity
δ_{ij}	Kronecker's delta
λ, μ	the Lamé elastic constants
ν	Poisson's ratio for the shell
ν_m	Poisson's ratio for the elastic medium
σ_{ij}	Trefftz components of stress
σ_{ij}^a	additional stresses in the elastic medium induced by the shell displacements
τ_{ij}	components of stress per unit original area shown in Fig. 2
ω	rotation in the linear theory of elasticity

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1. INTRODUCTION

STABILITY of equilibrium of a long cylindrical shell, which is uniformly compressed by a surrounding elastic medium, is examined by considering possible adjacent equilibrium states. The buckling load is then defined as the smallest load that admits a non-axially symmetric neighboring equilibrium configuration for a cylindrical shell, which initially has a perfectly circular cross-section. The initial equilibrium state is defined by the middle surface coordinate, $r = a$, and a uniform lateral surface pressure p. Since the displacements associated with the initial surface pressure are negligible in most applications, e.g. see [1], the position of initial equilibrium will also be called the undeformed position. The adjacent equilibrium state will be called the position of final equilibrium or the deformed position.

It is assumed that the medium is initially in equilibrium under an isotropic, homogeneous state of stress given by the pressure p. However, if p is the critical pressure, an adjacent equilibrium position exists and is defined by the tangential and radial middle surface shell displacements. Because the shell and the medium are in contact, the shell displacements induce additional surface tractions and produce changes in the initial surface pressure at the shell-medium interface. The resulting distributed force system on the shell surface in its deformed position is determined from the linearized theory for an elastic body under initial stress. These surface tractions are then related to the shell displacements, and the equations presented by Armenakas and Herrmann [2] are used to derive an expression for the critical pressure. This expression contains the elastic material constants of the shell and medium, the radius to thickness ratio of the shell, and the buckling mode number. In applying the buckling formula to determine the critical pressure, the mode number associated with the minimum pressure must be determined.

It is convenient in later Sections to refer to the two well-known related buckling problems of long cylindrical shells subjected to uniform hydrostatic and uniform constant-directional pressures. Although these loadings are the same for the shell in its undeformed position, they differ in the deformed position, e.g. see [3]. For the hydrostatic pressure, the direction of a surface traction on a deformed shell element is normal to the shell and its magnitude changes in proportion to the area change produced by deformation; whereas, for the constant-directional pressure, both the direction and magnitude per unit original surface area remain unchanged. The load which buckles a shell with Young's modulus E, Poisson's ratio ν, and radius to thickness ratio a/h is

$$p_0 = \frac{1}{4}\frac{E}{(1-\nu^2)}\left(\frac{h}{a}\right)^3 \qquad \text{for hydrostatic pressure}$$

and

$$p_c = \frac{1}{3}\frac{E}{(1-\nu^2)}\left(\frac{h}{a}\right)^3 \qquad \text{for constant-directional pressure.}$$

In both these examples the shell buckles in an inextensional mode; hence, the differences in the buckling loads result only from the change in the direction of the surface tractions for the case of the hydrostatic pressure.

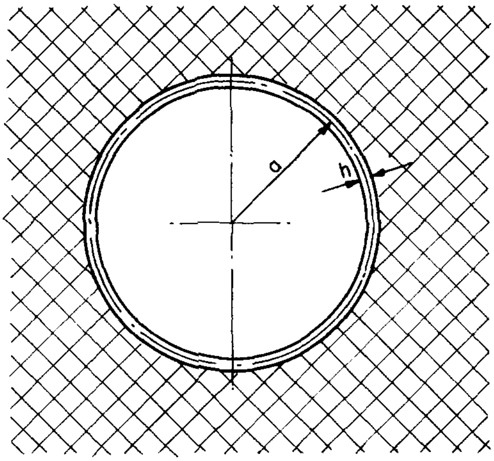


FIG. 1. Circular cylindrical shell surrounded by elastic medium.

Several limiting cases of the present problem are compared with the critical pressure of a shell loaded by hydrostatic and constant-directional pressures, and results of a more practical nature are presented for a range of shell and medium parameters.

2. SHELL EQUILIBRIUM EQUATIONS

Several linearized theories of motion for elastic, circular, cylindrical shells subjected to a general state of initial stress have been advanced by Herrmann and Armenakas [4]. In Ref. [2] these equations were applied to study the effect of several particular states of initial stress on the dynamic response of an infinitely long shell, i.e. the motion was independent of the axial coordinate. For a thin shell of radius a and thickness h, subjected to an initial state of uniform lateral pressure p, the equations in [2], for equilibrium, reduce to

$$(E_p - pa)\frac{\partial^2 v}{\partial \theta^2} + pav + (E_p - 2pa)\frac{\partial w}{\partial \theta} + a^2 \Delta F_\theta = 0, \tag{1a}$$

$$(E_p - 2pa)\frac{\partial v}{\partial \theta} + E_p w - pa\left(w - \frac{\partial^2 w}{\partial \theta^2}\right) + \frac{D}{a^2}\left(w + 2\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4}\right) - a^2 \Delta F_r - \frac{ha}{2}\frac{\partial}{\partial \theta}(\Delta F_\theta) = 0, \tag{1b}$$

where E_p is the shell compressional modulus, D is the shell flexural modulus, and v, w are the circumferential and radial middle surface displacements, measured positive clockwise and radially outward, respectively. The terms ΔF_θ, ΔF_r are the circumferential and radial components of the change due to deformation of the shell surface tractions, taken per unit undeformed middle surface area. In the notation of Ref. [4]

$$\Delta F_\theta = \Delta F_\theta^i + F_\theta^a,$$

$$\Delta F_r = \Delta F_r^i + F_r^a.$$

The changes ΔF_θ^i, ΔF_r^i are those related to the initial pressure, and F_θ^a, F_r^a are the additional shear and normal components of stress at the interface directly induced by the shell displacements.

For the problems discussed in [2] and [3] changes in the shell surface tractions were specified. In the present problem the values of ΔF_θ, ΔF_r are unknown *a priori* and must be determined by solving an appropriate boundary value problem for the surrounding elastic medium. Thus, ΔF_θ, ΔF_r will depend on the conditions at the shell–medium interface and the elastic constants of the surrounding medium.

3. EQUATIONS OF THE ELASTIC MEDIUM

The linearized equations for an elastic medium in the presence of initial stress are developed from the nonlinear theory of elasticity. Equilibrium equations and boundary conditions for the nonlinear theory, referred to a cylindrical coordinate system, can be obtained from the principle of virtual displacements, as was done by Novozhilov [5] with reference to a Cartesian system. For a state of plane strain the stress equilibrium equations are

$$\begin{aligned}&\sigma_{rr}\left(1+\frac{\partial u_r}{\partial r}+r\frac{\partial^2 u_r}{\partial r}\right)+\frac{\partial \sigma_{rr}}{\partial r}\left(r+r\frac{\partial u_r}{\partial r}\right)-\sigma_{\theta\theta}\left(1+\frac{u_r}{r}+\frac{2}{r}\frac{\partial u_\theta}{\partial \theta}-\frac{1}{r}\frac{\partial^2 u_r}{\partial \theta^2}\right)\\&\quad+\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)+2\sigma_{r\theta}\left(\frac{\partial^2 u_r}{\partial r\partial\theta}-\frac{\partial u_\theta}{\partial r}\right)+\frac{\partial \sigma_{r\theta}}{\partial r}\left(\frac{\partial u_r}{\partial \theta}-u_\theta\right)\\&\quad+\frac{\partial \sigma_{r\theta}}{\partial \theta}\left(1+\frac{\partial u_r}{\partial r}\right)=0,\end{aligned} \tag{2a}$$

$$\begin{aligned}&\sigma_{rr}\left(\frac{\partial u_\theta}{\partial r}+r\frac{\partial^2 u_\theta}{\partial r^2}\right)+\frac{\partial \sigma_{rr}}{\partial r}r\frac{\partial u_r}{\partial r}+\sigma_{\theta\theta}\left(\frac{1}{r}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)\\&\quad+\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\left(1+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right)+2\sigma_{r\theta}\left(1+\frac{\partial u_r}{\partial r}+\frac{\partial^2 u_\theta}{\partial r\partial\theta}\right)+\frac{\partial \sigma_{r\theta}}{\partial \theta}\frac{\partial u_\theta}{\partial r}\\&\quad+\frac{\partial \sigma_{r\theta}}{\partial r}\left(r+u_r+\frac{\partial u_\theta}{\partial \theta}\right)=0,\end{aligned} \tag{2b}$$

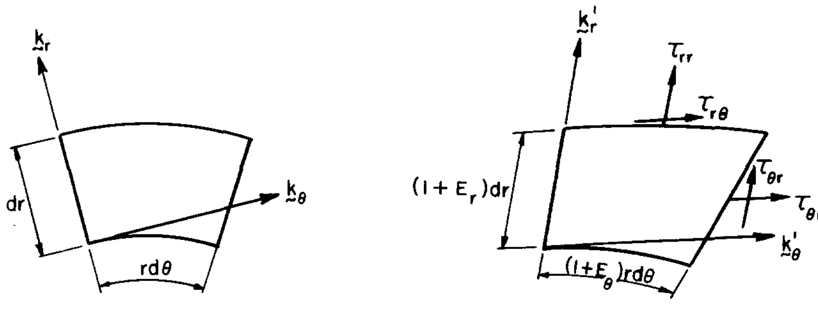
where u_θ, u_r are the displacement components and σ_{ij} are the Trefftz components of stress, which form a symmetric tensor. The physical significance of the Trefftz components of stress becomes apparent by considering a deformed elastic element, such as that shown in Fig. 2(b). If the force vector acting on the surface of a deformed element is resolved into non-orthogonal components parallel to the unit vectors $\mathbf{k}'_r, \mathbf{k}'_\theta$, the stress components τ_{ij} are defined as these force components divided by the face area before deformation. The Trefftz components of stress are then related to the matrix τ_{ij} by

$$\sigma_{ij}=\frac{\tau_{ij}}{1+E_j}$$

where E_j is the unit elongation of a line element originally in the j-direction.

The components of traction on the surface $r = a$, f_r, f_θ, taken parallel to the unit vectors $\mathbf{k}_r$, $\mathbf{k}_\theta$ (see Fig. (2(a)) and per unit undeformed surface area, are related to the Trefftz components of stress by

$$f_r=\left(1+\frac{\partial u_r}{\partial r}\right)\sigma_{rr}+\frac{1}{a}\left(\frac{\partial u_r}{\partial \theta}-u_\theta\right)\sigma_{r\theta}\qquad \text{at } r=a, \tag{3a}$$



(a) UNDEFORMED ELEMENT (b) DEFORMED ELEMENT

FIG. 2.

$$f_\theta = \left(1+\frac{u_r}{a}+\frac{1}{a}\frac{\partial u_\theta}{\partial \theta}\right)\sigma_{r\theta}+\frac{\partial u_\theta}{\partial r}\sigma_{rr} \qquad \text{at } r = a. \tag{3b}$$

Equations (2) and (3) can now be used to establish a set of linearized equations for an elastic body under high initial stresses subject to small additional disturbances. It is first assumed that the general deformed configuration is reached from an unstressed and unstrained state by passing through an intermediate equilibrium state, the state of initial stress. For application to the present problem it is specified that the elastic medium is in initial equilibrium under an isotropic, homogeneous state of stress given by the pressure p. The deformations associated with the initial pressure p are then neglected, and the medium is allowed to reach its final equilibrium position by assuming small deviations from the position of initial equilibrium. This final equilibrium state is defined by the additional displacements u_r, u_θ, which produce small strains and rotations, and the stresses $\sigma_{ij} = -p\delta_{ij}+\sigma^a_{ij}$. The linearized equilibrium equations are obtained by substituting $\sigma_{ij} = -p\delta_{ij}+\sigma^a_{ij}$ into equations (2) and neglecting all products of the additional stresses σ^a_{ij} and displacement gradients, but retaining all products of the initial pressure p and displacement gradients. Thus, the linearized equilibrium equations are

$$\frac{\partial \sigma^a_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma^a_{r\theta}}{\partial \theta}+\frac{\sigma^a_{rr}-\sigma^a_{\theta\theta}}{r}-p\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}\right) = 0, \tag{4a}$$

$$\frac{1}{r}\frac{\partial \sigma^a_{\theta\theta}}{\partial \theta}+\frac{2}{r}\sigma^a_{r\theta}+\frac{\partial \sigma^a_{r\theta}}{\partial r}-p\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}\right) = 0. \tag{4b}$$

Since the additional strains and rotations are small as compared to unity, it is assumed that the differences between the initial and final stresses are related to the additional strains in accordance with Hooke's law for an isotropic, elastic solid. Guided by the work of Biot [6], the differences between the initial pressure and the Trefftz components of stress σ_{ij} are taken to be proportional to the additional strains, i.e. σ^a_{ij} are taken proportional to the additional strains. Then, using the stress–displacement relations for small strains and rotations, e.g. see [7], the linearized equilibrium equations in terms of displacements may be written in the form

$$(\lambda+2\mu-p)\frac{\partial \Delta}{\partial r}-2(\mu-p)\frac{1}{r}\frac{\partial \omega}{\partial \theta} = 0, \tag{5a}$$

$$(\lambda+2\mu-p)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+2(\mu-p)\frac{\partial\omega}{\partial r} = 0, \tag{5b}$$

where λ, μ are the Lamé constants, Δ is the dilatation, and ω is the rotation. Thus, the initial pressure p merely has the effect of modifying the elastic constants; this is also pointed out in [8]. Furthermore, since p is usually much less than the values of λ, μ, the equilibrium equations governing the classical linear theory of elasticity may be employed to calculate the additional stresses and displacements.

The linearized tractions on the surface, $r = a$, are

$$f_r = -\left(1+\frac{\partial u_r}{\partial r}\right)p+\sigma_{rr}^a \qquad \text{at } r = a, \tag{6a}$$

$$f_\theta = -\frac{\partial u_\theta}{\partial r}p+\sigma_{r\theta}^a \qquad \text{at } r = a.$$

Since ΔF_θ and ΔF_r are defined as the circumferential and radial components of the change due to deformation of the shell surface tractions, taken per unit undeformed surface area, these components are

$$\Delta F_\theta = -\frac{\partial u_\theta}{\partial r}p+\sigma_{r\theta}^a \qquad \text{at } r = a, \tag{7a}$$

$$\Delta F_r = -\frac{\partial u_r}{\partial r}p+\sigma_{rr}^a \qquad \text{at } r = a. \tag{7b}$$

As will be shown, the changes ΔF_θ, ΔF_r can be related to the middle surface shell displacements, and the shell equilibrium equations (1) can then be used to formulate the buckling condition.

4. SHELL SURFACE TRACTIONS AT A BONDED INTERFACE

It has been shown in the previous Section that for the case of an elastic body subject to an initial state of isotropic, homogeneous stress, the linear theory of elasticity may be employed to determine the additional stresses and displacements. The shell is assumed bonded to the elastic medium and the middle surface shell displacements are taken as

$$v = V\sin n\theta, \tag{8a}$$

$$w = W\cos n\theta. \tag{8b}$$

Then the boundary conditions for the medium are

$$u_\theta = V\sin n\theta+\frac{nh}{2a}W\sin n\theta \qquad \text{at } r = a, \tag{9a}$$

$$u_r = W\cos n\theta \qquad \text{at } r = a, \tag{9b}$$

where the second term in equation (9a) represents the tangential displacement at the outer surface of the shell due to the rotation of the shell elements. Values of σ_{rr}^a, $\sigma_{r\theta}^a$ and $\partial u_r/\partial r$, $\partial u_\theta/\partial r$ at $r = a$ are now calculated from the field equations of classical elasticity for plane strain and the boundary conditions given by equations (9).

Following the procedure outlined in [9], the stress function

$$\phi = (C_1 r^{-n} + C_2 r^{-n+2}) \cos n\theta \qquad n \geqslant 2, \tag{10}$$

where C_1, C_2 are arbitrary constants, yields the following additional stresses and derivatives of the displacements at $r = a$

$$\sigma^a_{rr} = -\frac{E_m \cos n\theta}{a(1+\nu_m)}\left\{\left[\frac{2n(1-\nu_m)+(1-2\nu_m)}{3-4\nu_m}\right]W + \left[\frac{n(1-2\nu_m)+2(1-\nu_m)}{3-4\nu_m}\right]\left(V+\frac{nh}{2a}W\right)\right\}, \tag{11a}$$

$$\sigma^a_{r\theta} = -\frac{E_m \sin n\theta}{a(1+\nu_m)}\left\{\left[\frac{n(1-2\nu_m)+2(1-\nu_m)}{3-4\nu_m}\right]W + \left[\frac{2n(1-\nu_m)+(1-2\nu_m)}{3-4\nu_m}\right]\left(V+\frac{nh}{2a}W\right)\right\}, \tag{11b}$$

$$\frac{\partial u_r}{\partial r} = -\frac{\cos n\theta}{a}\left\{\left[\frac{2n(1-2\nu_m)+1}{3-4\nu_m}\right]W + \left[\frac{n+2(1-2\nu_m)}{3-4\nu_m}\right]\left(V+\frac{nh}{2a}W\right)\right\}, \tag{12a}$$

$$\frac{\partial u_\theta}{\partial r} = -\frac{\sin n\theta}{a}\left\{\left[-\frac{n-4(1-\nu_m)}{3-4\nu_m}\right]W + \left[\frac{4n(1-\nu_m)-1}{3-4\nu_m}\right]\left(V+\frac{nh}{2a}W\right)\right\}. \tag{12b}$$

It is now convenient to define the following dimensionless quantities

$$H = \frac{2n(1-\nu_m)+(1-2\nu_m)}{(1+\nu_m)(3-4\nu_m)}, \qquad G = \frac{n(1-2\nu_m)+2(1-\nu_m)}{(1+\nu_m)(3-4\nu_m)},$$

$$L = \frac{n-4(1-\nu_m)}{3-4\nu_m}, \qquad M = \frac{4n(1-\nu_m)-1}{3-4\nu_m},$$

$$R = \frac{2n(1-2\nu_m)+1}{3-4\nu_m}, \qquad S = \frac{n+2(1-2\nu_m)}{3-4\nu_m}. \tag{13}$$

Equations (7) then become

$$\Delta F_\theta = \left\{\frac{p}{a}\left[-LW + M\left(V+\frac{nh}{2a}W\right)\right] - \frac{E_m}{a}\left[GW + H\left(V+\frac{nh}{2a}W\right)\right]\right\} \sin n\theta, \tag{14a}$$

$$\Delta F_r = \left\{\frac{p}{a}\left[RW + S\left(V+\frac{nh}{2a}W\right)\right] - \frac{E_m}{a}\left[HW + G\left(V+\frac{nh}{2a}W\right)\right]\right\} \cos n\theta. \tag{14b}$$

Equations (14) give the changes of the surface tractions due to deformation in terms of the shell displacements, and the material constants of the medium.

5. STABILITY CONDITION FOR A BONDED INTERFACE

A formula for the critical pressure p can now be obtained from the shell equilibrium equations (1) and the expressions for ΔF_θ, ΔF_r calculated from the linearized theory of elasticity. Substitution of equations (8) and (14) into equation (1) yields

$$V\left\{n^2\frac{E_p}{a^2} + \frac{E_m}{a}H - \frac{p}{a}(n^2+1+M)\right\} + W\left\{\frac{nE_p}{a^2} + \frac{E_m}{a}\left(G+\frac{nh}{2a}H\right) - \frac{p}{a}\left(2n-L+\frac{nh}{2a}M\right)\right\} = 0, \tag{15a}$$

$$V\left\{\frac{nE_p}{a^2}+\frac{E_m}{a}\left(G+\frac{hn}{2a}H\right)-\frac{p}{a}\left(2n+S+\frac{nh}{2a}M\right)\right\}+W\left\{\frac{E_p}{a^2}+(n^2-1)^2\frac{D}{a^4}\right.$$
$$\left.+\frac{E_m}{a^2}\left[H\left(1+\frac{n^2h^2}{4a^2}\right)+\frac{nhG}{a}\right]-\frac{p}{a}\left[n^2+1+R+\frac{nh}{2a}(S-L)+\left(\frac{nh}{2a}\right)^2M\right]\right\}=0. \qquad (15b)$$

Equations (15) are linear homogeneous expressions, and the determinant of the coefficients of V, W must vanish for a nontrivial solution. Thus,

$$\left(\frac{p}{p_0}\right)^2\left\{(n^2-1)^2+(n^2+1)(R+M)+MR+LS+2n(L-S)+\frac{nh}{2a}[(n^2+1)(S-L)-4nM]\right.$$
$$\left.+(n^2+1)\left(\frac{nh}{2a}\right)^2M\right\}-\left(\frac{p}{p_0}\right)\left\{4\left(\frac{a}{h}\right)^2\left[(n^2-1)^2+n^2R+n(L-S)+M\right.\right.$$
$$\left.+\frac{nh}{2a}\left(n^2(S-L)-2nM+\frac{n^3h}{2a}M\right)\right]+\frac{(n^2-1)^2}{3}(n^2+1+M)+4\frac{E_m}{E}(1-v^2)\left(\frac{a}{h}\right)^3$$
$$\left.\times\left[H[2(n^2+1)+M+R]-G[4n+S-L]+\frac{nh}{2a}(n^2+1)\left(\frac{nh}{2a}H+2G\right)\right]\right\}$$
$$+\left\{\frac{4}{3}\left(\frac{a}{h}\right)^2n^2(n^2-1)^2+\frac{4}{3}\frac{E_m}{E}(1-v^2)\left(\frac{a}{h}\right)^3H(n^2-1)^2\right.$$
$$+16\frac{E_m}{E}(1-v^2)\left(\frac{a}{h}\right)^5\left[H(n^2+1)-2nF+\frac{nh}{2a}\left(\frac{n^3h}{2a}H+2n^2G-2nH\right)\right]$$
$$\left.+\left[4\frac{E_m}{E}(1-v^2)\left(\frac{a}{h}\right)^3\right]^2(H^2-F^2)\right\}=0 \qquad n\geqslant 2, \qquad (16)$$

where p_0 is the buckling pressure of a long cylindrical shell subjected to a uniform hydrostatic pressure and is given by

$$p_0=\frac{E}{4(1-v^2)}\left(\frac{h}{a}\right)^3. \qquad (17)$$

If the values of the nondimensional constants are substituted into equation (16), the formula becomes

$$\left(\frac{p}{p_0}\right)^2\left\{n(n+1)(n^2-1)+\frac{nh}{a}\left[n^2+1+\left(\frac{nh}{4a}(n^2+1)-2n\right)\left(\frac{4n(1-v_m)-1}{3-4v_m}\right)\right]\right\}$$
$$-\left(\frac{p}{p_0}\right)\left\{4\left(\frac{a}{h}\right)^2\left[(n^2-1)^2-2n+\frac{n^2-1+2n^3(1-2v_m)+4n(1-v_m)}{3-4v_m}+\frac{n^3h}{a}\right.\right.$$
$$\left.-\frac{n^2h}{2a}\left(2-\frac{n^2h}{2a}\right)\left(\frac{4n(1-v_m)-1}{3-4v_m}\right)\right]+\frac{(n^2-1)^2}{3}\left(n^2+1+\frac{4n(1-v_m)-1}{3-4v_m}\right)$$
$$+8\frac{E_m}{E}(1-v^2)\left(\frac{a}{h}\right)^3(n^2-1)\left(\frac{2(n+1)(1-v_m)-(1-2v_m)}{(1+v_m)(3-4v_m)}\right)$$
$$\left.+4\frac{E_m}{E}(1-v^2)\left(\frac{a}{h}\right)^3\frac{nh}{2a}(n^2+1)\left[\frac{nh}{2a}\left(\frac{2n(1-v_m)+(1-2v_m)}{(1+v_m)(3-4v_m)}\right)+\frac{2n(1-2v_m)+4(1-v_m)}{(1+v_m)(3-4v_m)}\right]\right\}$$

$$+\left\{\frac{4}{3}\left(\frac{a}{h}\right)^2 n^2(n^2-1)^2+\frac{4}{3}\left(\frac{E_m}{E}\right)(1-\nu^2)\left(\frac{a}{h}\right)^3(n^2-1)^2\left(\frac{2n(1-\nu_m)+(1-2\nu_m)}{(1+\nu_m)(3-4\nu_m)}\right)\right.$$

$$+16\frac{E_m}{E}(1-\nu^2)\left(\frac{a}{h}\right)^5(n^2-1)\left(\frac{2n(1-\nu_m)-(1-2\nu_m)+n^2h/a(1-2\nu_m)}{(1+\nu_m)(3-4\nu_m)}\right)$$

$$+16\frac{E_m}{E}(1-\nu^2)\left(\frac{a}{h}\right)^5\left(\frac{n^2h}{2a}\right)^2\left(\frac{2n(1-\nu_m)+(1-2\nu_m)}{(1+\nu_m)(3-4\nu_m)}\right)$$

$$\left.+\left[4\frac{E_m}{E}(1-\nu^2)\left(\frac{a}{h}\right)^3\right]^2\frac{n^2-1}{(1+\nu_m)^2(3-4\nu_m)}\right\}=0 \qquad n\geqslant 2. \tag{18}$$

Substitution of a practical range of values into equation (18) indicates that one root of the quadratic equation is much smaller than the other. Then the smaller root may be obtained with sufficient accuracy by neglecting the quadratic term in equation (18). A further simplification is achieved by noting that throughout a practical range of parameters several terms are so small as to be negligible in applications. A simple yet accurate expression for obtaining the critical pressure of a shell surrounded by an elastic medium is given by ($n \geqslant 2$)

$$\frac{p}{p_0}=\frac{[n^2(n^2-1)(3-4\nu_m)]/3+4\dfrac{E_m}{E}\left(\dfrac{a}{h}\right)^3\dfrac{(1-\nu^2)}{(1+\nu_m)}\left[2n(1-\nu_m)+(1-2\nu_m)\left(\dfrac{n^2h}{a}-1\right)\right]}{(n^2-1)(3-4\nu_m)+2n(1-2\nu_m)+1}. \tag{19}$$

The critical pressure is seen to depend on the mode number n, and in each application the mode number n associated with the minimum value of p/p_0 must be determined.

It was assumed in the analysis that the shell buckled elastically. To ensure that the yield stress of the shell material has not been exceeded, the hoop stress corresponding to the buckling pressure must be calculated.

It has been shown that the changes due to deformation of the initial surface pressure ΔF_θ^i, ΔF_r^i depend on the material constants ν_m and the mode number n, e.g. see equations (7) and (12). It is of interest to calculate the critical pressure of the mode, $n = 2$, when the additional stresses σ_{rr}^a and $\sigma_{r\theta}^a$ are neglected, and then compare these results with the critical pressures corresponding to constant-directional and hydrostatic loadings. Since p_0 is the buckling pressure of a long cylindrical shell due to hydrostatic pressure

$$\frac{p}{p_0}=1 \qquad \text{for hydrostatic pressure}$$

and

$$\frac{p}{p_0}=\frac{4}{3} \qquad \text{for constant-directional pressure.}$$

As pointed out in the Introduction, the change in direction of the surface loads for the hydrostatic pressure gives rise to the difference of these critical pressures. These examples are now compared with some limiting cases of the present problem.

By letting E_m approach zero and setting $n = 2$, equation (19) gives

$$\frac{p}{p_0}=1 \qquad \text{for } \nu_m = 0{\cdot}5$$

and

$$\frac{p}{p_0} = \frac{6}{7} \qquad \text{for } \nu_m = 0.$$

6. APPROXIMATE STABILITY CONDITION FOR A SMOOTH INTERFACE

For the bonded case the same material elements at the shell–medium interface, which are in contact before deformation, remain in contact after deformation. Thus, changes in the shell surface tractions due to deformation could be directly related to the shell displacements. If, however, the shell–medium interface is smooth, i.e. shear stresses between the shell and the medium are absent, points at the interface are permitted to move relative to each other, and the changes in the shell surface tractions cannot be directly related to the shell displacements. This is a consequence of the linearized theory of elasticity which differentiates between the geometry of an element in the medium before and after deformation.

A solution to the problem with a smooth interface using the linearized theory to represent the medium appears to be complicated and will be approximated by the classical linear theory of elasticity. This theory does not account for the change due to deformation of the initial surface pressure. Thus, $\Delta F_\theta^i = \Delta F_r^i = 0$, which corresponds to the constant-directional loading, and equations (7) reduce to

$$\Delta F_\theta = 0 \qquad \text{at } r = a, \tag{20a}$$

$$\Delta F_r = \sigma_{rr}^a \qquad \text{at } r = a. \tag{20b}$$

The value of σ_{rr}^a may be calculated from the stress function ϕ given by equation (10), and the boundary conditions

$$u_r = W \cos n\theta \qquad \text{at } r = a, \tag{21a}$$

$$\sigma_{r\theta}^a = 0 \qquad \text{at } r = a. \tag{21b}$$

Following the procedure outlined in [9]

$$\Delta F_r = \sigma_{rr}^a = \frac{-E_m W \cos n\theta}{a(1+\nu_m)}\left[\frac{n^2-1}{(1-2\nu_m)(n+1)+n}\right]. \tag{22}$$

Substitution of equations (8) and (22) into the shell equilibrium equations (1) leads to

$$\frac{p}{p_0} = \frac{n^2\left[\dfrac{n^2-1}{3} + 4K\left(\dfrac{1-\nu^2}{E}\right)\left(\dfrac{a}{h}\right)^3\right]}{(n^2-1)\left[\dfrac{1}{12}\left(\dfrac{h}{a}\right)^2 + 1\right] + K\left(\dfrac{1-\nu^2}{E}\right)\dfrac{a}{h}} \qquad n \geqslant 2. \tag{23}$$

For thin shells the quantity $1/12\,(h/a)^2$ may be neglected when compared to unity and

$$\frac{p}{p_0} = \frac{n^2\left[\dfrac{n^2-1}{3} + 4K\left(\dfrac{1-\nu^2}{E}\right)\left(\dfrac{a}{h}\right)^3\right]}{n^2 - 1 + K\left(\dfrac{1-\nu^2}{E}\right)\left(\dfrac{a}{h}\right)} \qquad n \geqslant 2, \tag{24}$$

where

$$K = \frac{E_m}{1+\nu_m}\left[\frac{1}{(1-2\nu_m)(n+1)+n}\right]. \tag{25}$$

Still another approximate formula can be derived if the initial surface tractions are prescribed to act as a hydrostatic pressure, i.e. the traction vector on a deformed shell element is specified to remain normal to the shell surface and to change magnitude in proportion to the area change produced by deformation. Taking the values of ΔF_θ^i, ΔF_r^i for the case of a hydrostatic pressure from [2] and adding σ_{rr}^a at $r = a$, given by equation (22)

$$\Delta F_\theta = -\frac{p}{a}(V+nW)\sin n\theta, \tag{26a}$$

$$\Delta F_r = -\left[\frac{p}{a}(nV+W)+\frac{K}{a}(n^2-1)W\right]\cos n\theta. \tag{26b}$$

Substitution of equations (26) into the shell equilibrium equations (1) leads to the formula

$$\frac{p}{p_0} = \frac{n^2-1}{3}+4K\left(\frac{1-\nu^2}{E}\right)\left(\frac{a}{h}\right)^3 \qquad n \geqslant 2. \tag{27}$$

If the medium has a Poisson's ratio $\nu_m = 0.5$, equation (27) is identical to equation (19) which is the solution for a bonded interface. Thus, for a medium with $\nu_m = 0.50$ the additional shear stresses at the interface are negligible, i.e. the magnitude of $\sigma_{r\theta}^a$ at $r = a$ is not large enough to influence the buckling pressure of the shell.

Examination of the exact buckling formula for the case of a bonded interface, equation (16), provides an indication of the range of accuracy for the approximate formulae, equations (24) and (27). The parameters L, M pertain to the changes in direction of the initial surface tractions, while R, S pertain to the magnitude changes. The dominant terms in equation (16) containing these parameters, as can be verified by direct substitution, are the terms in the linear coefficient given by

$$4\left(\frac{a}{h}\right)^2[(n^2-1)^2+n^2R+n(L-S)+M].$$

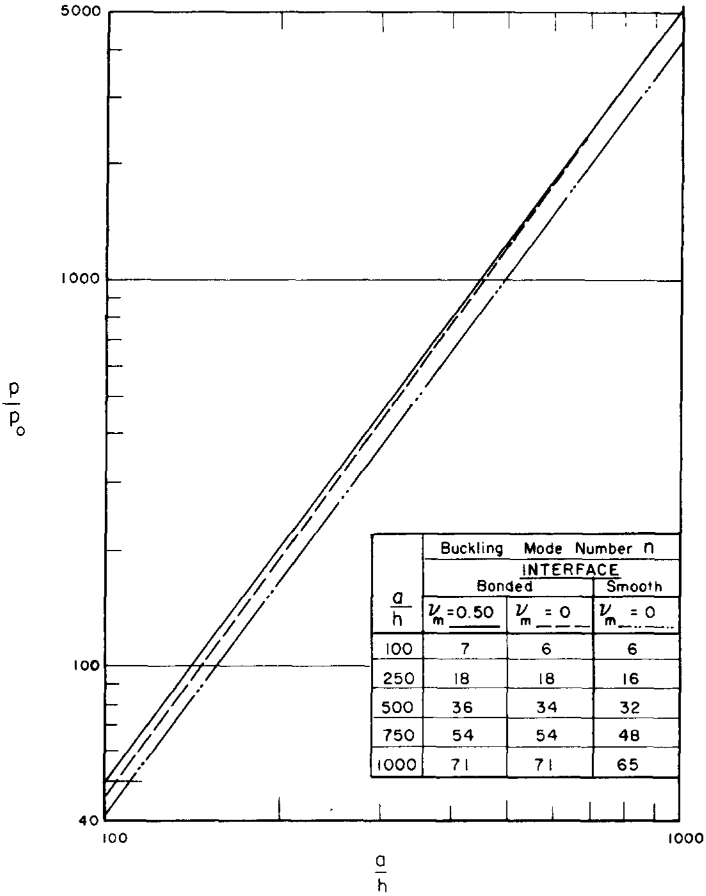
Substitution of L, M, R, S for $\nu_m = 0$ into this expression gives

$$4\left(\frac{a}{h}\right)^2\left[(n^2-1)^2+\frac{2n}{3}(n^2-1)+\frac{1}{3}(n^2-1)\right],$$

where the second term in the above bracket results from combining R, S and the third from combining L, M. Then, if n is large as compared to 2/3, the approximate formulae for the case of a smooth interface should yield accurate results.

7. NUMERICAL EXAMPLE

Values of the critical pressure and mode number are presented in Fig. 3 for $E/E_m = 10^4$ and a/h ranging from 100 to 1000. As pointed out in the previous Section, when $\nu_m = 0{\cdot}50$ the results for the bonded and smooth interfaces coincide. For $\nu_m = 0$ results for the



$\frac{a}{h}$	Buckling Mode Number n — INTERFACE — Bonded: ν_m = 0.50	Bonded: ν_m = 0	Smooth: ν_m = 0
100	7	6	6
250	18	18	16
500	36	34	32
750	54	54	48
1000	71	71	65

FIG. 3. Critical pressure versus radius to thickness ratio for $E/E_m = 10^4$, $\nu = 0{\cdot}30$.

case of a smooth interface were calculated from the approximate equations (24) and (27) and almost identical values were obtained from the two expressions. Also, since over the range considered in Fig. 3 most wave numbers are large as compared to 2/3, the approximate results for the case of a smooth interface should be reasonably accurate. This estimated accuracy, as pointed out in Section 6, is based on a comparison with the exact solution for the bonded interface.

The hoop stress corresponding to the critical pressure p must also be calculated to ensure that the yield stress of the shell material is not exceeded. For a steel shell with a radius to thickness ratio $a/h = 100$ and a surrounding elastic medium with the properties $E/E_m = 10^4$, $\nu_m = 0{\cdot}5$, the buckling pressure is 420 psi and the corresponding hoop stress is 42,000 psi. If Young's modulus for the medium is reduced to $E/E_m = 10^5$, the buckling pressure is 89 psi and the corresponding hoop stress is 8920 psi.

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Zusammenfassung—Ein langer, dünner, kreisförmiger und zylindrischer Mantel steht unter einem uniformen Druck, und zwar wird dieser von dem elastischen Umgebungsmedium ausgeübt. Die Gleichgewichtsstabilität des Mantels wird untersucht, indem die möglichen benachbarten Gleichgewichtszustände in Betracht gezogen werden. Die durch das elastische Medium verursachte Beanspruchung des deformierten Mantels wird gefunden, indem ein zugehöriges Grenzwertproblem der linearisierten Elastizitätstheorie beim Vorhandensein einer anfänglichen Spannung gelöst wird. Für die Fälle einer anhaftenden und einer glatten Zwischenfläche werden Ausdrücke abgeleitet, welche den kritischen Druck angeben.

Абстракт—Длинная, тонкая, круглая цилиндрическая оболочка подвергнута равномерному внешнему давлению, вызванному окружающей упругой средой. Устойчивость равновесия оболочки исследуется посредством допущения возможных соседних состояний равновесия. Нагрузка, вызванная упругой средой, на оболочке в деформированном состоянии находится посредством ассоциированной проблемы граничной величины линейно преобразованной теории упругости в присутствии начального напряжения. Выводятся выражения, дающие критическое давление для случаев сцепления и гладкости поверхности раздела.